The dynamics of developing Cooper pairing at finite temperatures

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We study the time evolution of a system of fermions with pairing interactions at a finite temperature. The dynamics is triggered by an abrupt increase of the BCS coupling constant. We show that if initially the fermions are in a normal phase, the amplitude of the BCS order parameter averaged over the Boltzman distribution of initial states exhibits damped oscillations with a relatively short decay time. The latter is determined by the temperature, the single-particle level spacing, and the ground state value of the BCS gap for the new coupling. In contrast, the decay is essentially absent when the system was in a superfluid phase before the coupling increase.

Considerable progress has been made over the past few years in understanding the dynamical fermionic pairing in response to fast perturbations[1–9]. Recent interest in this long-standing problem[10–12] has been motivated by experiments on cold atomic fermions with tunable interactions[13, 14], even though other systems have also been considered[15, 16].

The general picture that emerged from the theory work is that as a result of the perturbation, e.g. a sudden change of the coupling constant, the system of fermions with pairing interactions can reach a variety of dynamical phases with properties quite distinct from the equilibrium ones [7–9]. For example, a steady state characterized by undamped periodic oscillations of the time dependent Bardeen-Cooper-Schrieffer (BCS) order parameter $\Delta(t)$ [1,8] and a gapless steady state[8, 9], $\Delta(t)=0$, have been identified.

Periodic oscillations occur in particular when at $t=0$ the fermions are described by a many-body wave function with a seed gap Δ_i much smaller than the ground state gap Δ_0. As a result of the Cooper instability of the initial state the order parameter starts to grow exponentially, $\Delta(t)=\Delta_i e^{\Delta_0 t}$, and reaches the ground state value at time $\tau/2=\ln(\Delta_0/\Delta_i)/\Delta_0$. However, in the absence of the energy relaxation the system does not equilibrate and it can be shown that $|\Delta(t)|$ is periodic in time with a period τ[1, 8].

In this Letter we study the effect of temperature fluctuations on the non-adiabatic dynamics of fermions with attractive interaction[6]. Suppose the system is initially in equilibrium at a finite temperature T. At $t=0$ the dynamics is triggered by an abrupt increase of the pairing strength and a certain quantity is measured at a later time. This process is repeated many times for each data point as is typical for measurements in atomic gases[17, 18]. We are therefore interested in dynamical quantities averaged over the Boltzman distribution of initial states.

Our main results are as follows. We show that, if before the coupling increase the system is in a normal phase at temperature T, the average amplitude of the order parameter, $\langle|\Delta(t)|\rangle$, displays exponentially damped oscillations with a decay time (see also Fig. 1)

$$\frac{t_0}{\langle\tau\rangle}=\frac{1}{\pi^2}\ln\left(\frac{4\Delta_0^2}{T\delta}\right) \tag{1}$$

where $\langle\tau\rangle$ is the average oscillation period and δ is the single particle level spacing. Here and below we assume $\delta\ll T\ll\Delta_0$. Expression (1) is accurate up to a prefactor of order one under the logarithm.

For typical values for cold atomic fermions[13, 14] Eq. (1) yields $t_0/\langle\tau\rangle\sim 1-3$, i.e. there are only a few regular oscillations before the dephasing sets in. In contrast, for the paired initial phase, we demonstrate that $t_0\propto 1/\sqrt{\delta}$ indicating that the decay time effectively diverges as the temperature is decreased below the critical temperature of the initial phase.

We emphasize that each time the coupling is switched a particular initial condition is selected and the system goes into a state with periodic $|\Delta(t)|$. However, whether the oscillations are seen in an ensemble averaged measurement depends on the quantity being measured. For example, it seems difficult to observe many of them in $\langle|\Delta(t)|\rangle$. On the other hand, since the fluctuations of the oscillation frequency are small[6] (see also below Eq. (13)), it can in principle be obtained e.g. from the ensemble averaged radio frequency absorption spectra[19].

The decay time (1) can be qualitatively understood as follows. In the normal state a nonzero initial value of the order parameter Δ_i is due to fluctuations, which in mesoscopic samples are governed by an energy scale $\sqrt{T\delta}$ [20, 21]. Changing Δ_i by a factor of order one in the expression for the period $\Delta_0\tau=2\ln(\Delta_0/\Delta_i)$ leads to changes in the period $\Delta_0\delta\tau\sim 1$. Then, one expects the average of $|\Delta(t)|$ over all possible values of Δ_i to dephase after $\tau/\delta\tau$ oscillations, i.e. on $t_0\propto\ln^2(\Delta_0/\sqrt{T\delta})/\Delta_0$ timescale. Note that the average period $\langle\tau\rangle\propto\ln(\Delta_0/\sqrt{T\delta})/\Delta_0$ and oscillation frequency remain finite. In the superfluid state the order parameter has a macroscopic thermal average $\bar{\Delta}_i$, while typical thermal fluctuations $\sqrt{T\delta}\ll\bar{\Delta}_i$. In this case repeating the above argument, we obtain $\Delta_0\delta\tau\sim\sqrt{T\delta}/\bar{\Delta}_i$ and

$\Delta_0 t_0 \sim \bar{\Delta}_i \ln^2(\Delta_0/\bar{\Delta}_i)/\sqrt{T\delta}$, i.e. an extremely long decay time.

The non-stationary Cooper pairing at times much shorter than the energy relaxation time can be described by the BCS model

$$\hat{H} = \sum_{j;\sigma=\downarrow,\uparrow} \epsilon_j \hat{c}^\dagger_{j\sigma}\hat{c}_{j\sigma} - \lambda\delta \sum_{j,k} \hat{c}^\dagger_{j\uparrow}\hat{c}^\dagger_{j\downarrow}\hat{c}_{k\downarrow}\hat{c}_{k\uparrow}, \qquad (2)$$

where ϵ_j are the single fermion energies relative to the Fermi level, δ is the mean spacing between ϵ_j, λ is the dimensionless BCS coupling constant, and $\hat{c}_{j\sigma}$ are the fermionic annihilation operators.

In the time-dependent BCS mean-field approach[1] the many-body wave function is a product state

$$|\Psi(t)\rangle = \prod_{n_m=0,2} \left(u_m(t) + v_m(t) c^\dagger_{m\uparrow}\hat{c}^\dagger_{m\downarrow} \right) |0\rangle, \qquad (3)$$

where $u_m(t)$ and $v_m(t)$ are the Bogoliubov amplitudes and the product is taken only over unoccupied ($n_m = 0$) and doubly occupied ($n_m = 2$) levels. Singly occupied levels are excluded since their occupation numbers are conserved by the Hamiltonian (2).

The time evolution of the system is governed by the Bogoliubov-de Gennes equations

$$i\dot{u}_m = \epsilon_m u_m + \Delta v_m, \quad i\dot{v}_m = -\epsilon_m v_m + \Delta^* u_m, \qquad (4)$$

where $\Delta = \lambda\delta \sum_m u_m v_m^*$. These equations can be cast into the form of equations of motion for classical spins[1]

$$\dot{\mathbf{s}}_m = 2\mathbf{b}_m \times \mathbf{s}_m, \quad \mathbf{b}_m = (-\Delta_x, -\Delta_y, \epsilon_m), \qquad (5)$$

where Δ_x and $-\Delta_y$ are the real and imaginary parts of $\Delta = \lambda\delta\sum_m s_m^-$ and the components of spins are related to Bogoliubov amplitudes u_m and v_m as follows

$$2s_m^z = |v_m|^2 - |u_m|^2, \;\; s_m^- \equiv s_m^x - i s_m^y = u_m v_m^*. \qquad (6)$$

For example, according to Eqs. (3,6) the Fermi ground state where all states below the Fermi level are occupied and states above are empty corresponds to $s_m^z = -\mathrm{sgn}\,\epsilon_m/2$ and $s_m^- = 0$.

Remarkably, nonlinear systems (4) and (5) turn out to be integrable[4]. The solution for $\Delta(t)$ can be obtained with the help of the Lax vector technique [7] by introducing

$$\mathbf{L}(w) = -\frac{\mathbf{z}}{\lambda\delta} + \sum_m \frac{\mathbf{s}_m}{w - \epsilon_m}, \qquad (7)$$

where w is an auxiliary parameter and $\mathbf{z}$ is a unit vector along the z axis. The square of the Lax vector is conserved by Eq. (5) for any w and therefore the roots of $\mathbf{L}^2(w) = 0$ are integrals of motion. Further, one can show that the majority of the roots lie on continuous lines, while the remaining isolated roots uniquely determine the form of $|\Delta(t)|$ at times $t \gg 1/\Delta_0$[7]. For instance, for initial states close to the Fermi ground state there are two isolated roots, $w_1 = i\gamma_1$ and $w_2 = i\gamma_2$, in the upper half plane of complex w. In this case the solution of Eq. (5) is known to be [1,4,6,7,8]

$$|\Delta(t)| = \Delta_+ \mathrm{dn}\left(\Delta_+(t - \tau/2), k\right), \quad k^2 = 1 - \frac{\Delta_-^2}{\Delta_+^2}, \quad (8)$$

where dn is the Jacobi elliptic function with modulus k, τ is its period, and $\Delta_\pm = |\gamma_1 \pm \gamma_2|$. Eq. (8) describes periodic in time $|\Delta(t)|$ whose period and amplitude are controlled by $\Delta_\pm$.

First, consider a Fermi gas at a temperature T and zero BCS coupling constant. At $t = 0$ the coupling is suddenly turned on so that $\Delta_0 \gg T$, where Δ_0 is the ground state gap for the new coupling. Before the interaction switch on the system can be in any eigenstate of the free Fermi gas with the probability given by its Boltzman weight. These states thus provide an ensemble of initial conditions for equations of motion (5) and our task is to evaluate the average of $|\Delta(t)|$ over all possible initial states.

In the non-interacting problem amplitudes (u_m, v_m) take values $(0,0)$, $(1,0)$, and $(0,1)$ corresponding to occupancies $n_m = 1$, 0, and 2, respectively. Note that they are always correlated so that $s_m^- = u_m v_m^* = 0$ and $s_m^z = \pm 1/2$ or 0 indicating that the eigenstates of the free Fermi gas are (unstable) stationary states for the mean-field equations of motion (4,5). However, for any nonzero coupling they are not exact stationary states of the quantum Hamiltonian (2) before the mean-field decoupling of the interaction term. These quantum effects facilitate the development of the Cooper instability and after a short time states of the form (3) with finite $u_m v_m^*$ can be used. In the spin language, the spins $\mathbf{s}_m$ acquire nonzero s_m^-, i.e. nonzero components in the xy plane.

As argued in Ref. 6 only spins at energies $|\epsilon_m| \lesssim T \ll \Delta_0$ initially have appreciable xy components (see below). It follows from Eq. (7) that $\mathbf{L}^2(w)$ has two isolated roots in the upper half plane of complex w and the order parameter is described by Eq. (8), where the parameters $\Delta_\pm$ are

$$\Delta_+ \approx \Delta_0, \quad \Delta_- \approx 2\delta \left| \sum_m s_m^- \right|. \qquad (9)$$

The values of s_m^- are random with a distribution determined by the Boltzman distribution of initial states and the quantum effects discussed above. On the other hand, there is a large number $N \sim T/\delta$ of random complex numbers in the sum (9) and as noted in Ref. 6 (see Eq. (46) therein) one therefore expects the Rayleigh distribution[22]

$$p(\Delta_-) = C\Delta_- \exp\left(-\frac{\alpha\Delta_-^2}{4T\delta} \right) \qquad (10)$$

independent of the details of the distribution of s_m^-. Here $\sqrt{T\delta}$ is a characteristic scale of fluctuations of Δ_-, α is of order one, and C is a normalization constant.

Thus, averaging $|\Delta(t)|$ over Boltzman distributed initial states reduces to integrating Eq. (8) with respect to Δ_- with distribution (10), i.e.

$$\frac{\langle|\Delta(t)|\rangle}{\Delta_0} = \int_0^\infty \mathrm{dn}\,(\Delta_0(t-\tau/2), k)\, p(\Delta_-) d\Delta_- \qquad (11)$$

Note that the Jacobi function dn depends on Δ_- through its modulus $k = 1 - \Delta_-^2/\Delta_0^2$. For example, its period for $\Delta_- \ll \Delta_0$ is [23]

$$\tau = \frac{2}{\Delta_0} \ln\left(4\frac{\Delta_0}{\Delta_-}\right). \qquad (12)$$

Using Eqs. (12,10), we evaluate the average oscillation period and its standard deviation (see also Ref. 6),

$$\langle\tau\rangle = \frac{1}{\Delta_0}\ln\frac{4\Delta_0^2}{T\delta}, \quad \delta\tau = \frac{\pi}{\sqrt{6}}\frac{1}{\Delta_0}, \qquad (13)$$

up to a factor of order one under the logarithm. The average frequency and its deviation are $\langle\omega\rangle = 2\pi/\langle\tau\rangle$ and $\delta\omega/\langle\omega\rangle = \delta\tau/\langle\tau\rangle$.

The asymptotic behavior of integral (11) at large times $t \gg t_0$ can be evaluated using the saddle point method

$$\frac{\langle|\Delta(t)|\rangle}{\Delta_0} = \frac{1}{\sqrt{\Delta_0 t_0}} - \frac{4\sqrt{\Delta_0 t}}{\Delta_0 t_0} e^{-t/t_0} \cos[\eta(t)], \qquad (14)$$
$$\eta(t) = \frac{2t}{\pi t_0}\ln\frac{2t}{\pi t_0} - \frac{2t}{\pi t_0} + \frac{2\Delta_0 t}{\sqrt{\Delta_0 t_0}} + \frac{\pi}{4},$$

where t_0 is given by Eq. (1). We see that on the t_0 time scale $\langle|\Delta(t)|\rangle$ exponentially approaches a constant value smaller than the ground state gap Δ_0 by a large factor $\ln(\alpha\Delta_0^2/T\delta)/\pi$. The approach is oscillatory with a typical period close to the ensemble averaged period $\langle\tau\rangle$.

Next, we present several alternative systematic derivations of Eq. (10) and show that it is independent of the details of initial state distribution. First, note that Eqs. (5) are equations of motion for classical spin Hamiltonian $H = \sum_m 2\epsilon_m s_m^z - \lambda\delta\sum_{m,n} s_m^+ s_n^-$. As discussed above, before the interaction switch on the spins are along the z axis. Their z components take values $s_m^z = \pm 1/2$ or 0 with independent probabilities proportional to the corresponding Boltzman weight $e^{-2\epsilon_m s_m^z/T}$. This presents a technical difficulty, since these spin configurations are (unstable) equilibria for Eqs. (5).

One way to circumvent this problem is to replace the above ensemble of initial spin configurations with the Boltzman distribution of classical spins of length $s_m = 1/2$. Then, each spin $\mathbf{s_m}$ is characterized by polar and azimuthal angles θ_m and ϕ_m with independent probability proportional to $\exp(-\epsilon_m\cos\theta_m/T)$, i.e. spins at $|\epsilon_m| \lesssim T$ acquire finite components in the xy plane. Using this probability distribution and Eq. (9), we evaluate $p(\Delta_-)$. The calculation results in Eq. (10) with $\alpha = 2/\ln(\Delta_0/T)$ and we obtain Eqs. (1,13) and (14).

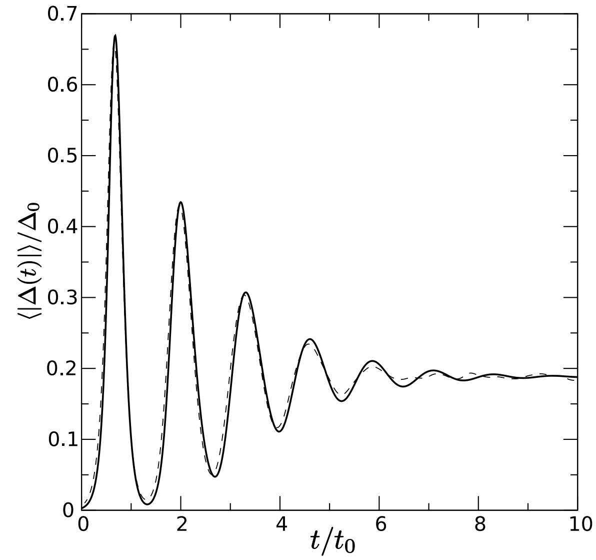

FIG. 1: Time evolution of the amplitude of the BCS order parameter $|\Delta(t)|$ averaged over initial states at temperature T. Numerical simulation of Eq. (5) for 10^4 spins averaged over 10^4 realizations of initial conditions (15) (solid curve) is compared to expression (11) (dotted curve). The time is in units of the decay time t_0, Eq. (1); the ground state gap is $\Delta_0 = 2\times 10^3\delta$ and $T = 400\delta$, where δ is the level spacing.

A distribution of the form (10) for $|\Delta(t=0)|$ was obtained in Ref. 6 for an ensemble of initial conditions suggested in the same reference. Note that according to Eq. (3) $|u_m|^2$ and $|v_m|^2$ represent probabilities of zero and double occupancy, respectively, of the level ϵ_m. In the free Fermi gas before the interaction is turned on their thermal averages are $\langle|u_m|^2\rangle = n_m^2$ and $\langle|v_m|^2\rangle = (1-n_m)^2$, where $n_m = (e^{\epsilon_m/T}+1)^{-1}$ is the Fermi function. Averaging Eq. (4) with respect to ϵ_m in a narrow window of energies, we can replace u_m and v_m with $\langle u_m\rangle = e^{i\epsilon_m t} n_m^2$ and $\langle v_m\rangle = e^{-i\epsilon_m t} e^{i\phi_m}(1-n_m)^2$, where ϕ_m is a random relative phase. Since the total energy of the free gas does not depend on ϕ_m, they are assumed to have independent uniform distributions. Further, assuming $\langle u_m v_m^*\rangle \approx \langle u_m\rangle\langle v_m^*\rangle$, and using Eq. (6), we obtain the following initial spin configurations [6]

$$s_m^z = -\frac{1}{2}\tanh\left(\frac{\epsilon_m}{2T}\right), \quad s_m^- = \frac{e^{-i\phi_m}}{4\cosh^2\left(\frac{\epsilon_m}{2T}\right)} \qquad (15)$$

Using Eqs. (15, 9) and uniform distributions for ϕ_m, we derive Eqs. (1,13) and Eq. (10) with $\alpha = 6$, see also Fig. 1.

Finally, Eqs. (1,13) and (14) can be derived starting from the Ginzburg-Landau free energy. The advantage of this approach is that we can consider initial states with nonzero BCS coupling that is suddenly increased at $t = 0$. The ground state gap for the new coupling, Δ_0, is assumed to be much larger than that for the old coupling. Then, the equation $\mathbf{L}^2(w) = 0$ has two isolated roots with $\operatorname{Im} w > 0$ and the evolution of the order parameter is

described by Eq. (8) as before. With the help of Eq. (7), we obtain $\Delta_+ \approx \Delta_0$ and $\Delta_- \approx 2\Delta_i \ln(\Delta_0/\Delta_i)$, where Δ_i is the gap for the old coupling.

First, consider the case $T > T_c$, where T_c is the critical temperature for the old coupling. To calculate the average of $|\Delta(t)|$ over initial states, we need the probability distribution of Δ_- or equivalently the distribution of possible values of the gap Δ_i before the coupling increase. We assume the latter is of the Ginzburg-Landau form $\Delta_i \exp(-F(\Delta_i)/T)$, where the free energy for $T > T_c$ is $F(\Delta_i) = \ln(T/T_c)|\Delta_i|^2/\delta$ [20, 21]. Using this distribution function and the above expressions for $\Delta_\pm$ in terms of Δ_i, we again obtain Eq. (14), where now

$$t_0 \approx \frac{1}{\pi^2 \Delta_0} \ln^2\left(\frac{\ln(T/T_c)\Delta_0^2}{T\delta}\right), \tag{16}$$

This expression holds for $T - T_c \gg \sqrt{T_c\delta}$.

Below the critical temperature, for $T_c > T_c - T \gg \sqrt{T_c\delta}$, we keep the quartic term in $F(\Delta_i)$ and expand Eq. (11) in $\Delta_i - \bar{\Delta}_i$, where $\bar{\Delta}_i$ is the thermal average of the order parameter before the coupling change. Using a saddle point method, we obtain a Gaussian decay to a constant value on $t_0 \propto \bar{\Delta}_i^2 \ln^2(\Delta_0/\bar{\Delta}_i)/\sqrt{T^3\delta}$ timescale. On the other hand, the dynamics at times this long is likely not described by the Hamiltonian (2) that does not account for energy relaxation. Thus, we see that the dephasing of ensemble averaged oscillations due to thermal fluctuations is effectively absent when the dynamics is started in the paired phase. The reason is that in this case the order parameter has a macroscopic initial average much larger than its thermal fluctuations. The fast dephasing above T_c crosses over into a slow dephasing below T_c in a narrow window of temperatures $|T - T_c| \sim \sqrt{T_c\delta}$.

In conclusion, we studied the effect of thermal fluctuations on the dynamics of fermions with pairing interactions triggered by an abrupt increase of the pairing strength. We showed that if the system is in the normal phase before the coupling increase, the amplitude of the order parameter averaged over the Boltzman distribution of initial states exhibits damped oscillations with relatively short decay time (1), see Eq. (14). On the other hand, the damping is essentially absent when the dynamics starts from the superfluid phase.

An interesting problem is to determine the time evolution described by the *quantum* Hamiltonian (2) at $T = 0$ starting from the Fermi ground state, i.e. the ground state of the Hamiltonian (2) for $\lambda = 0$. Extending the above considerations to this case, one might expect damped oscillations due to quantum fluctuations *without* ensemble averaging. If this is the case, an estimate for the decay time can be obtained by replacing the temperature T in Eq. (1) with the level spacing δ, i.e. $t_0 \sim \ln^2(\Delta_0/\delta)/\Delta_0$.

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